The NJL Model for Quarks in Hadrons and Nuclei
Part III: Nuclear Matter, Quark Matter and Neutron Stars

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NJL for nuclear systems

- The NJL model is **very simple**, and works well to describe properties of single hadrons.
- Traditional nuclear physics treats nucleons as point particles. But are there **"quark effects" in the nucleus**?
- We know that in nuclear systems there are **strong mean fields**, mainly a **scalar** (attractive) and a **vector** (repulsive) mean field. Quarks inside the nucleons feel these mean fields: **Origin of "medium modifications"**. Examples: Modification of electromagnetic form factors measured in proton knock-out $(e, e'p)$ reactions; Modification of structure functions measured in deep inelastic electron-nucleus scattering ($\Rightarrow$ **EMC effect**). **NJL model is suitable to describe these phenomena!**
- Is there a **phase transition** from nuclear matter to to quark matter at high densities? ($\Rightarrow$ relevant for **neutron stars**).

Nuclear matter: Mean fields

In nuclear matter, the quarks feel a **scalar potential** (incorporated into the mass M), and a **vector potential** (called V^μ).

- Follow our earlier **mean field description** of the vacuum: Start from the NJL Lagrangian, including the vector interaction term $-G_\omega\left(\overline{\psi}\gamma^\mu\psi\right)^2$. Add $\left(-M\overline{\psi}\psi - V_\mu\overline{\psi}\gamma^\mu\psi + C\right)$, and subtract again. Assume

$$\overline{\psi}\psi = \langle\overline{\psi}\psi\rangle + :\overline{\psi}\psi:\,, \qquad \overline{\psi}\gamma^\mu\psi = \langle\overline{\psi}\gamma^\mu\psi\rangle + :\overline{\psi}\gamma^\mu\psi:$$

  where $\langle\overline{\psi}\psi\rangle$ and $\langle\overline{\psi}\gamma^\mu\psi\rangle$ now refer to **nuclear matter**.

- Require that $\mathcal{L}_{\rm res}$ has **no c-number terms** and **no terms linear** in $:\overline{\psi}\psi:$ and $:\overline{\psi}\gamma^\mu\psi:$. This gives

$$\begin{aligned} M &= m - 2G_\pi\langle\overline{\psi}\psi\rangle\,, \qquad V^\mu = 2G_\omega\langle\overline{\psi}\gamma^\mu\psi\rangle \\ C &= -\frac{(M-m)^2}{4G_\pi} + \frac{V^2}{4G_\omega} \end{aligned}$$

Note: For nuclear matter at rest, only V^0 is nonzero ($V^i = 0$).

Nuclear matter energy density

Next, write down an expression for the **energy density**.
What is the difference between the vacuum and nuclear matter?
The presence of nucleons! They have a **mass** $M_N(M)$ determined from the Faddeev equation; they feel a **vector potential** $3V^0$; and they move with momenta up to the **Fermi momentum** p_F. (Baryon density is $\rho = 2p_F^3/(3\pi^2)$.) Therefore,

$$\mathcal{E}(M) = \mathcal{E}_{\rm vac}(M) - \frac{V_0^2}{4G_\omega} + 4\int^{p_F} \frac{\mathrm{d}^3k}{(2\pi)^3}\left(\sqrt{M_N(M)^2 + k^2} + 3V^0\right)$$

Note: By including the mean vector field V^0 in the Faddeev equation, one can confirm that the nucleon energy in the medium is $\epsilon_p = \sqrt{M_N(M)^2 + k^2} + 3V^0$.

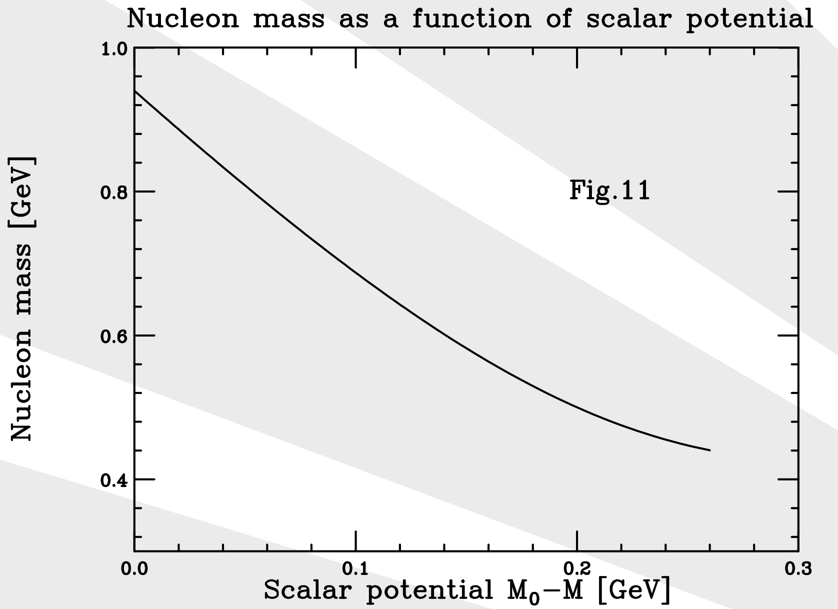
Finally, M and V^0 are determined by **minimization**: For fixed p_F,

$\partial\mathcal{E}/\partial M = 0 \Rightarrow$ **in-medium gap equation**
$\partial\mathcal{E}/\partial V^0 = 0 \Rightarrow V^0 = 6G_\omega\rho$.

The function $M_N(M)$

The function $M_N(M)$ is obtained from the Faddeev equation (or its static approximation).

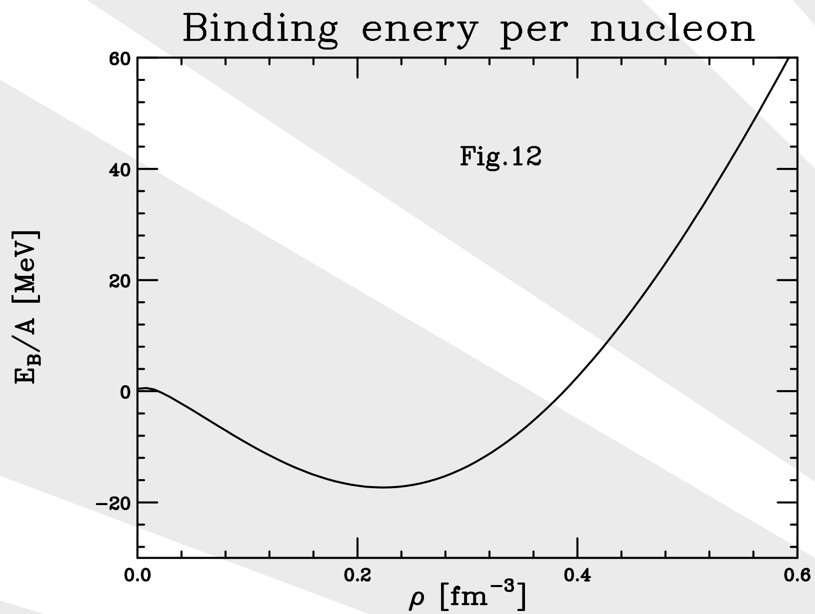
So far, we only needed $M_N(M_0$=0.4 GeV) = 0.94 GeV, but the in-medium gap equation will give solutions $M < M_0$ for finite density.



Note that there is a curvature (“**scalar polarizability**”), which is important for saturation of the binding energy per nucleon in nuclear matter.

Binding energy per nucleon

Binding energy per nucleon: $E_B/A = \mathcal{E}/\rho - M_{N0}$, where M_{N0} = nucleon mass in vacuum = 0.94 GeV:

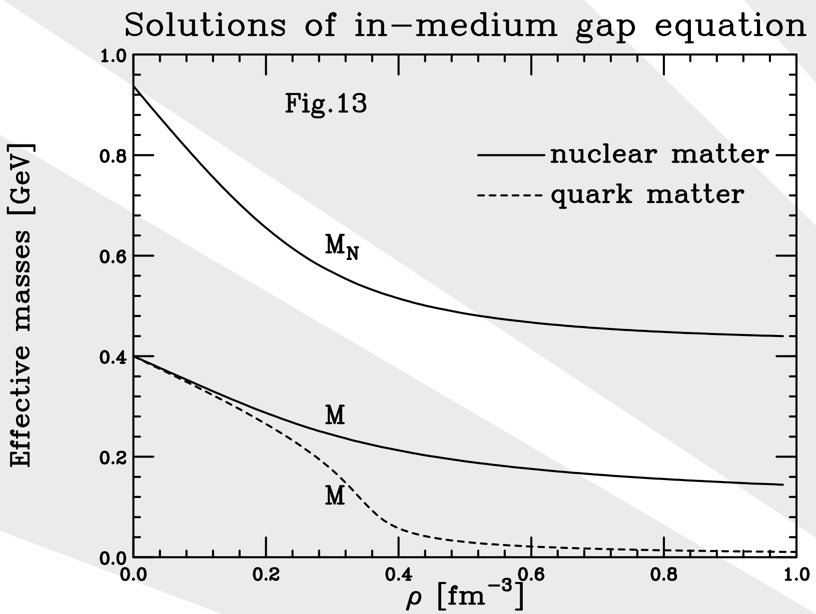


The strength of the vector mean field (G_ω) is adjusted so that the curve passes through the empirical saturation point $(E_B/A, \rho)$=(-15 MeV, 0.16 nucleons/ fm^{-3}).

Important for saturation: Unphysical quark thresholds for nucleon are absent in the proper-time regularization scheme.

Solutions of in-medium gap equation

Nucleon and quark masses as functions of density:



The dashed line shows the quark mass in quark matter for comparison (to be discussed later), and indicates a chiral phase transition at relatively low densities.

In nuclear matter, no strong indication of chiral restoration is seen.

Application 5: The EMC effect (1)

In 1982, the **European Muon Collaboration (EMC)** observed that the structure function of the nucleus (F_{2A}) is not equal to the sum of free nucleon structure functions. Many calculations have shown that this **cannot** be explained by binding and Fermi motion of nucleons. Is this a **"medium modification"** of the single nucleon structure function?

In the parton model,

$$F_{2A}(x) = x \sum_q e_q^2 \, f_q^A(x)$$

where $0 < x < 1$ is the Bjorken variable for the nucleus, and $f_q^A(x)$ is the

- **probability that quark q has (light cone) momentum fraction x in the nucleus A;**
- or: (probability that nucleon has fraction y in nucleus) $\times$ (probability that quark has fraction x/y in the nucleon).

The EMC effect (2)

Mathematically, this can be expressed by a convolution:

$$f_q^A(x) = \int_0^1 \mathrm{d}y \int_0^1 \mathrm{d}z\, \delta(x - yz)\, f_q^N(z)\, f_N^A(y)$$

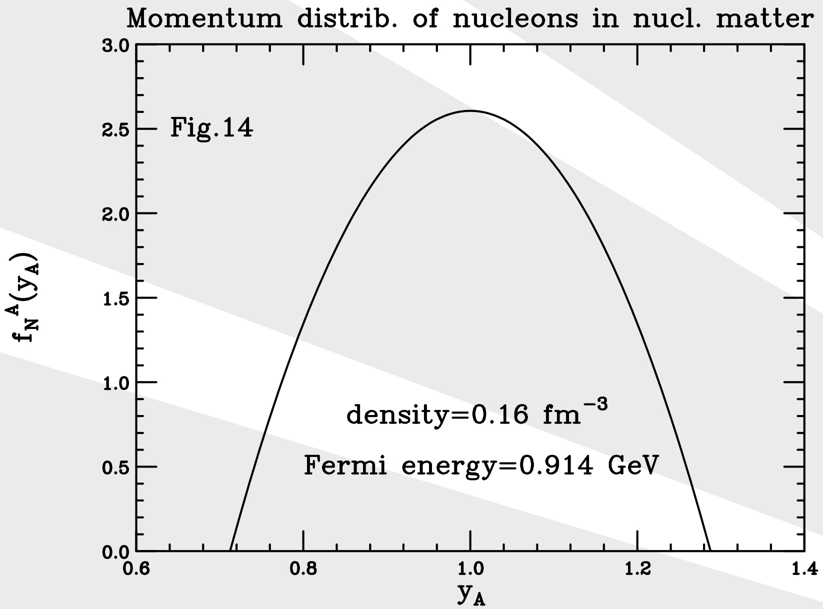
- We know already how to calculate $f_q^N(z)$. Remember: It is the **quark 2-point function (propagator) inside the nucleon, for fixed quark momentum component** $k^+ = p^+ z$.
- Then we also know how to calculate $f_N^A(y)$: It is the **nucleon propagator in nuclear matter, for fixed nucleon momentum component** $p^+ = yP^+$. (Here $P^+ = (P^0 + P^3)/\sqrt{2} = M_A/\sqrt{2}$ refers to the total system at rest.)

Expect: $f_N^A(y)$ peaks at $y \simeq 1/A$, and $f_q^A(x)$ peaks at $x \simeq 1/(3A)$. To avoid small numbers x, y, one usually uses $x_A \equiv Ax$ and $y_A \equiv Ay$. Then $f_N^A(y_A)$ will peak around $y_A \simeq 1$, and x_A around $x_A \simeq 1/3$.

Momentum distribution of nucleons

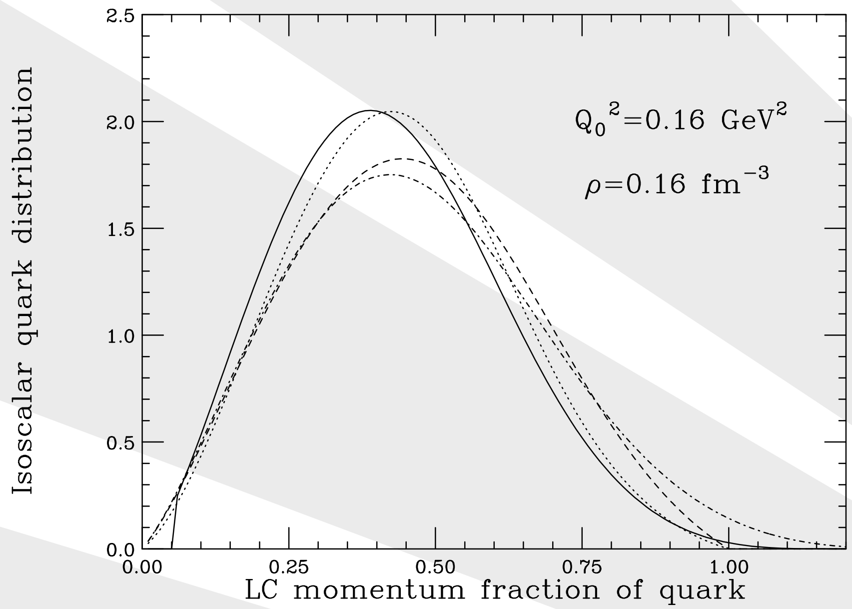
Momentum distribution of nucleons (per nucleon) in nuclear matter:

$$f_N^A(y_A) = \frac{3}{4}\left(\frac{\epsilon_F}{p_F}\right)^3 \left[\left(\frac{p_F}{\epsilon_F}\right)^2 - (1-y_A)^2\right]$$



Quark distribution in medium

Results for the **quark momentum distribution** (per nucleon) in isospin symmetric nuclear matter (sum of up and down quark distributions): **Fig.15**

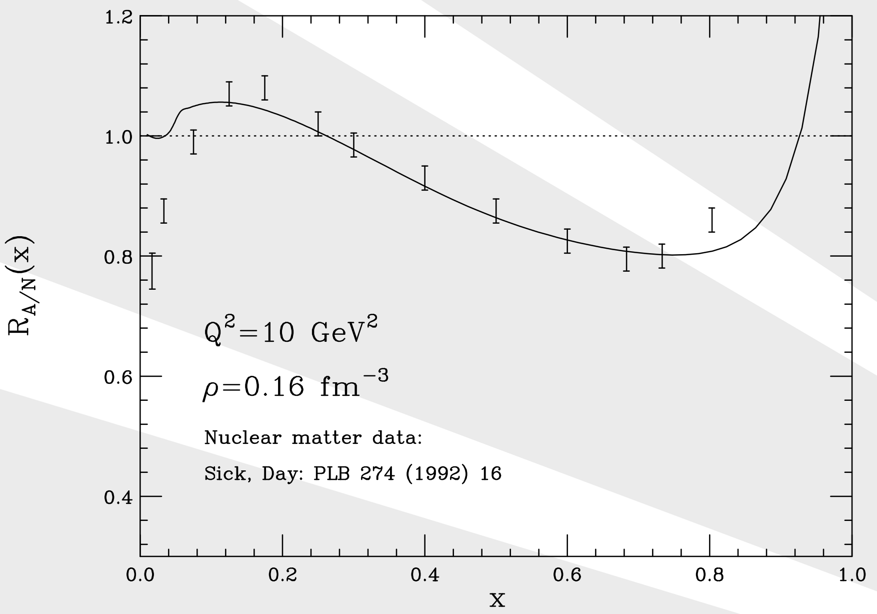


- dotted line . . . distribution in free nucleon
- dashed line . . . with in-medium masses
- dash-dotted line . . . incl. Fermi motion of nucleons
- solid line . . . total result, incl. effect of mean vector field.

EMC ratio in nuclear matter

"EMC ratio" in isospin symmetric nuclear matter: **Fig.16**

$$R_{A/N}(x) \equiv \frac{F_{2A}(x_A)}{ZF_{2p}(x) + NF_{2n}(x)} \overset{\text{parton model}}{\Longrightarrow} \frac{x_A f_q^A(x_A)}{x f_q^N(x)}$$



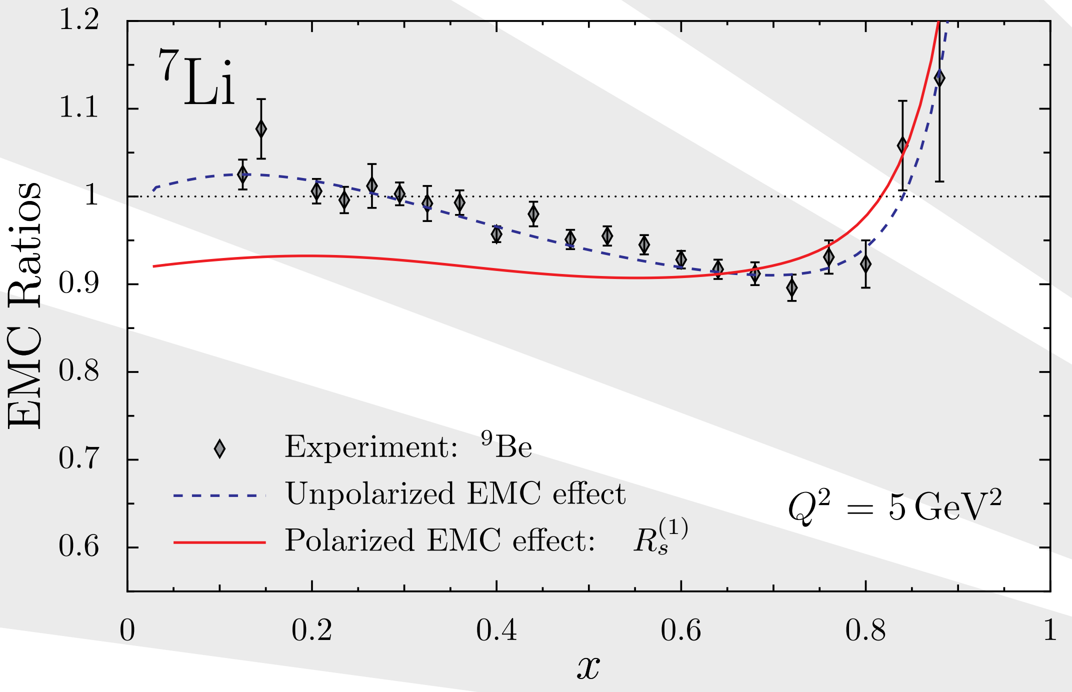
Note: In the figure, x is the Bjorken variable for the free nucleon.

Then $\frac{x_A}{x} = \frac{M_N}{\overline{M}_N}$, where M_N is the free nucleon mass, and $\overline{M}_N = M_A/A \gtrsim 1$ is the mass of the system per nucleon.

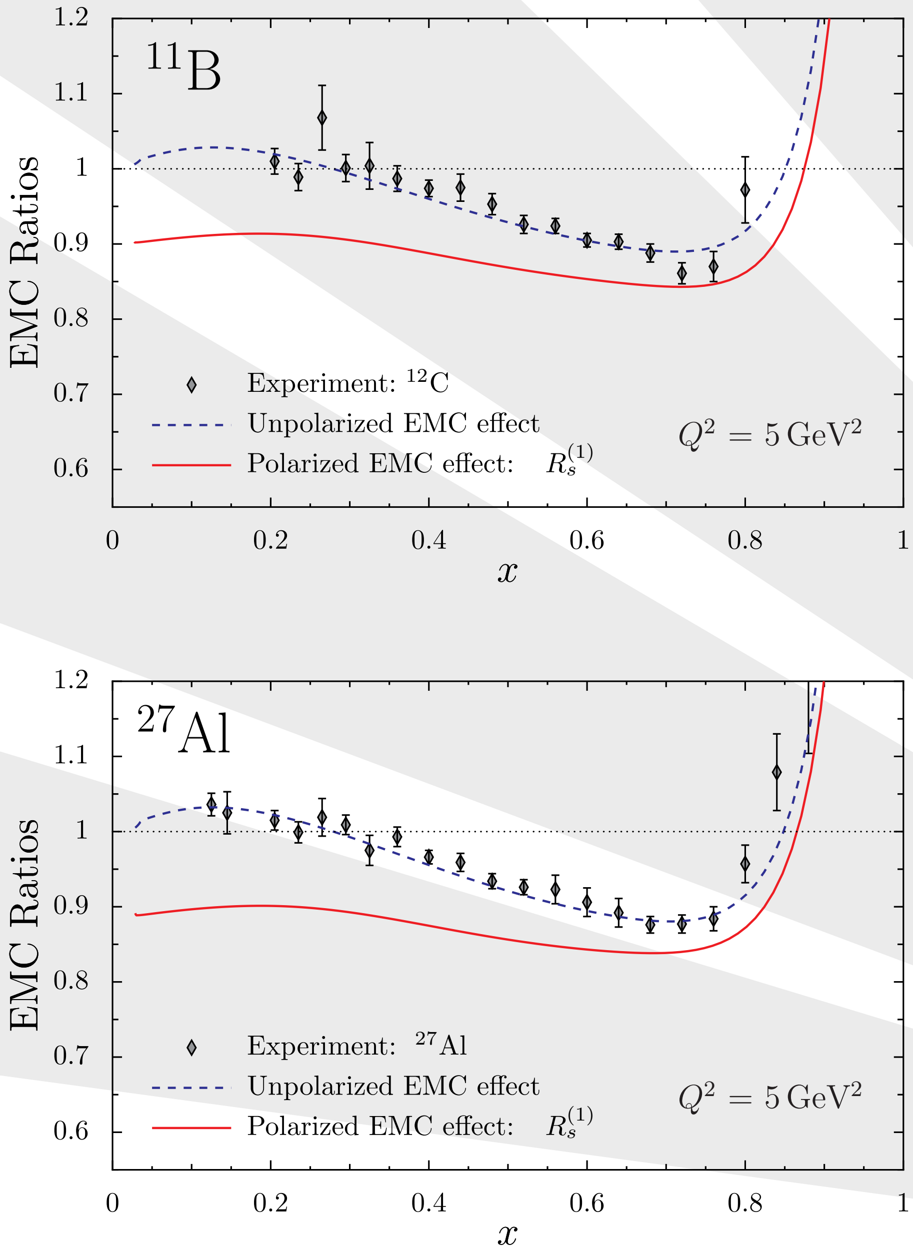
EMC effect in finite nuclei (1)

These calculations can be done also for **finite nuclei**, although we do not go into details here. We show examples for the nuclei ^{7}Li, ^{11}B and ^{27}Al. For these nuclei, **a new “polarized EMC effect” has been predicted.** We also show these exciting predictions, which will be tested at JLab experiments.

^{7}Li: **Fig.17**



EMC effect (2): Fig. 18



Equation of state at high densities

Using the NJL model, we can describe the saturation properties of nuclear matter. But **what happens at very high densities**, like in the interior of neutron stars?

- Many people believe that a **transition to quark matter** takes place.
- The NJL model has been used extensively to describe quark matter. In particular, the importance of a **color superconducting state** has been emphasized: The interaction in the scalar diquark channel gives rise to **pairing**, like in the BCS theory.
- Here we first construct the equation of state for quark matter, and then use the **Gibbs conditions** to look for **phase transitions** from nuclear matter to quark matter.

Quark matter

If we replace the nucleon Fermi motion in our expression for $\mathcal{E}$ by the quark Fermi motion

$$4\int^{p_F}\frac{\mathrm{d}^3k}{(2\pi)^3}\left(\sqrt{M_N^2+k^2}+3V^0\right)\to 12\int^{p_F}\frac{\mathrm{d}^3k}{(2\pi)^3}\left(\sqrt{M^2+k^2}+V^0\right)$$

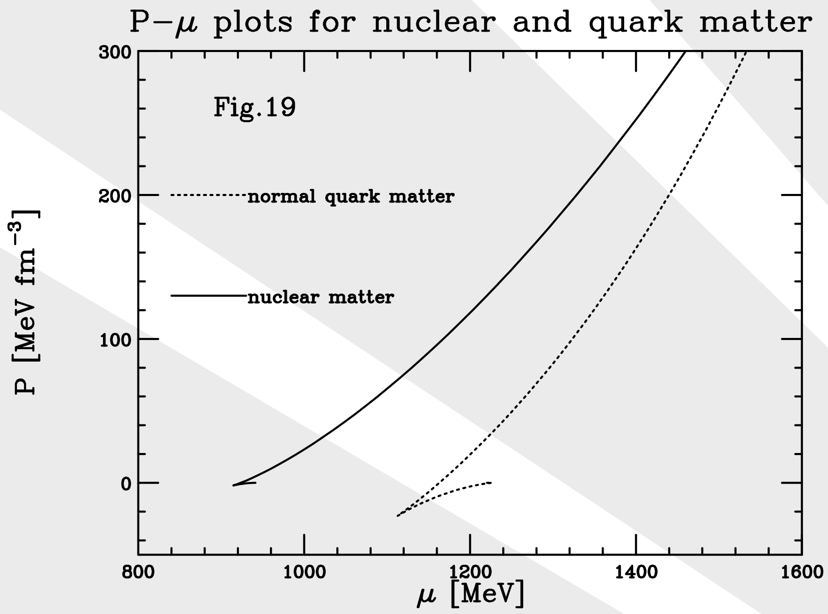
we can describe **quark matter** (at the same baryon number density $\rho = 2p_F^3/(3\pi^2)$). Eventually, we have **2 separate equations of state: Nuclear matter and quark matter**.
The **Gibbs condition** (for $T=0$) says that the phase with larger pressure (P) for given chemical potential (μ) is the stable phase:

$$P_{\text{stable}}(\mu) > P_{\text{unstable}}(\mu)$$

where

$$\begin{aligned} P &= \rho^2\frac{\partial}{\partial\rho}\left(\frac{\mathcal{E}}{\rho}\right) \\ \mu &= \frac{\partial\mathcal{E}}{\partial\rho} \end{aligned}$$

Phase transitions (1)



There is no crossing of the curves! This would mean that **nuclear matter is always the stable phase.**

Some details on these plots:

- The density increases along the lines, starting with $\rho = 0$ at the open ends.
- At low densities, there is a gas-liquid phase transition in the nuclear matter phase, and a chiral phase transition in the quark matter phase.
- For example, in the nuclear matter phase, for densities below the saturation density (where P crosses zero), the state is a mixture of "vacuum" and "nuclear matter": Nuclear matter droplets surrounded by vacuum.

Color superconductivity

So far, our classical fields were $\langle\overline{\psi}\psi\rangle$ (chiral condensate) and $\langle\overline{\psi}\gamma^0\psi\rangle$ (quark density). However, there is also the possibility of a non-zero **"diquark condensate"**

$$\Delta = -G_s\langle\overline{\psi}i\gamma_5\beta_2C\tau_2\overline{\psi}^T \pm \psi^T C^{-1}\tau_2 i\gamma_5\beta_2\psi\rangle$$

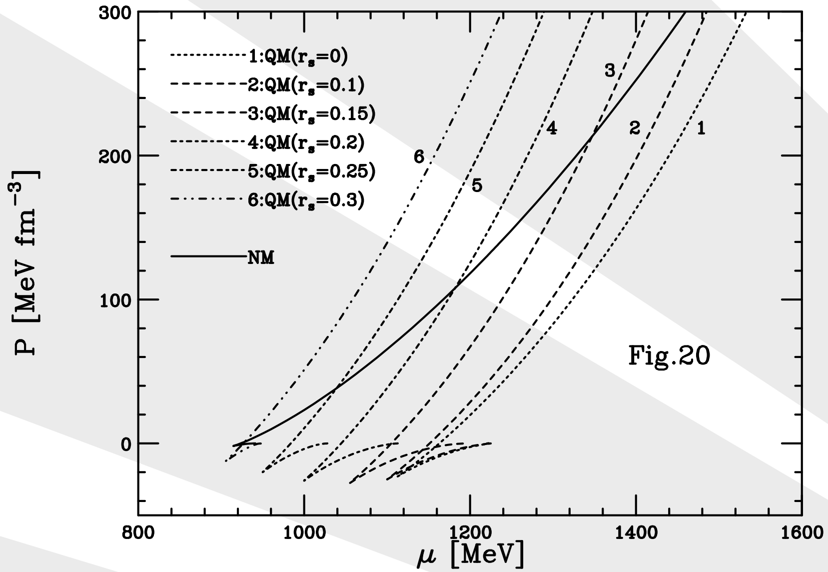
which corresponds to the gap in BCS theory.
If Δ is non-zero, the color symmetry is spontaneously broken $SU(3) \to SU(2)$ (because of the choice of β_2 among 3 possible diquark colors). Also the phase symmetry $U(1)$ is broken.

The mean field approximation with the 3 fields $\langle\overline{\psi}\psi\rangle$, $\langle\overline{\psi}\gamma^0\psi\rangle$, and Δ is done most conveniently in the **Nambu-Gorkov formalism**, using the quark field $\Psi = \frac{1}{\sqrt{2}}\left(\psi, C\tau_2\overline{\psi}^T\right)$.

Phase transitions (2)

By increasing the strength of the pairing interaction ($r_s = G_s/G_\pi$), quark matter becomes more stable, and a **phase transition** from nuclear matter to quark matter **becomes possible**:

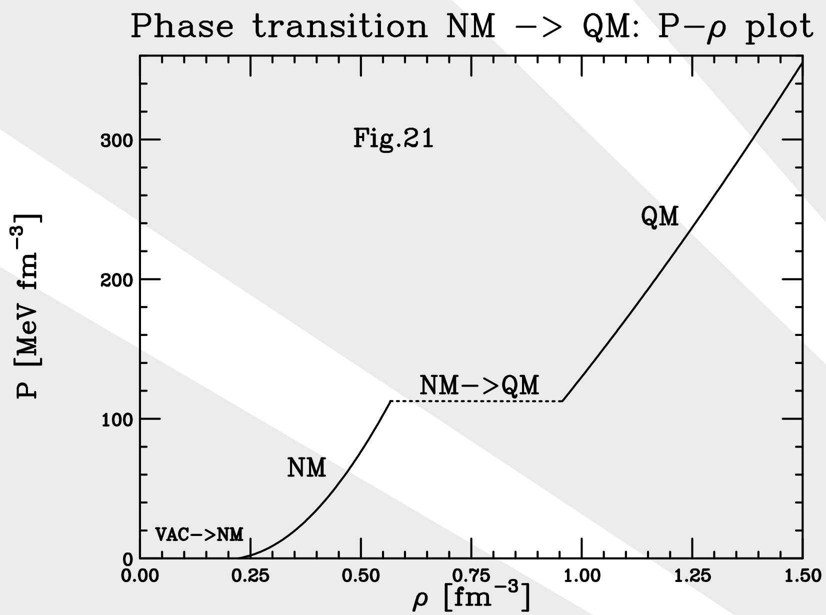
P−μ plots for nuclear and quark matter

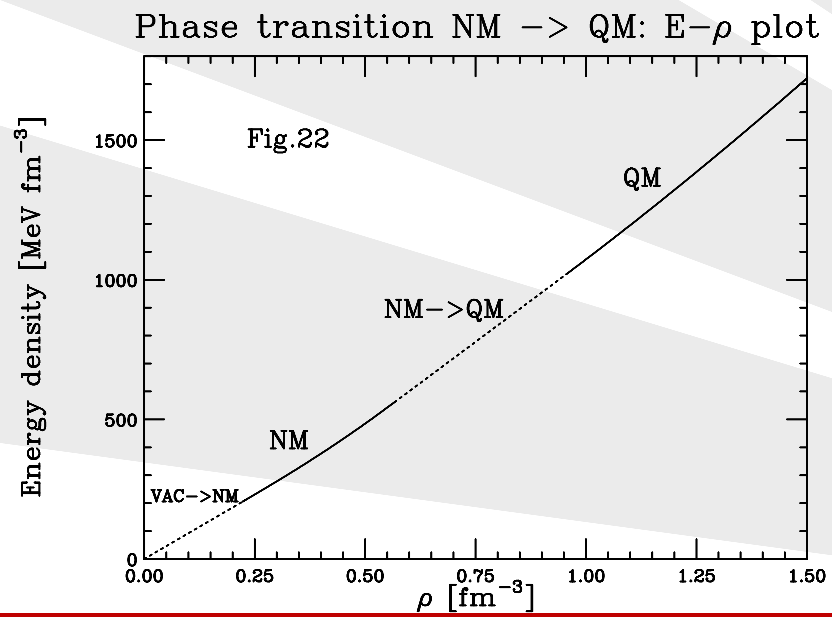


Here NM = nuclear matter, QM = quark matter.

Take $r_s = 0.2$ as an example: There is a **1st order phase transition** from nuclear to quark matter, which begins at $\rho = 0.57$ fm^{-3} (density at crossing on the NM curve) and ends at $\rho = 0.95$ fm^{-3} (density at crossing on the QM curve), see next slide.

Phase transitions (3)



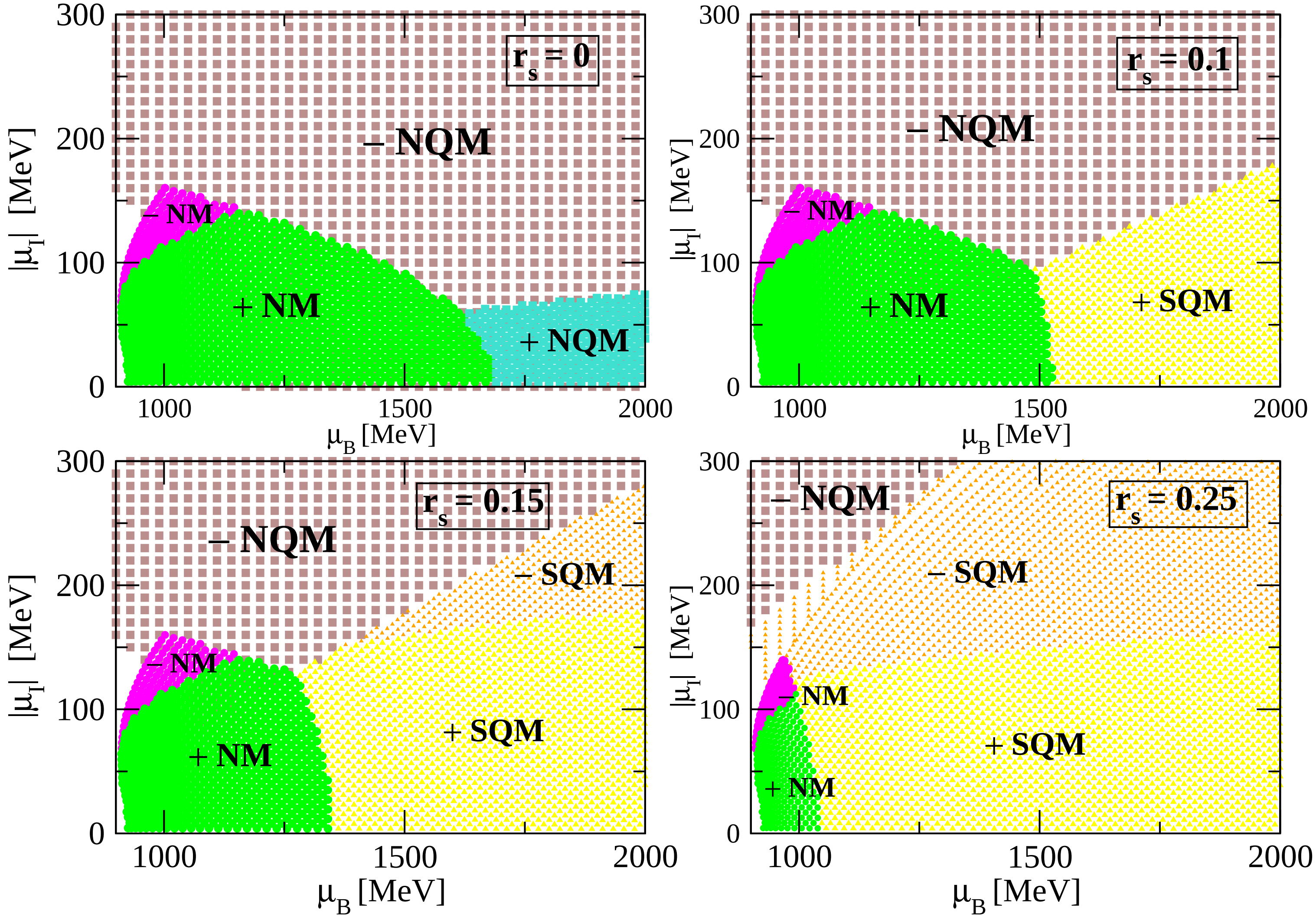


Application 6: Compact stars

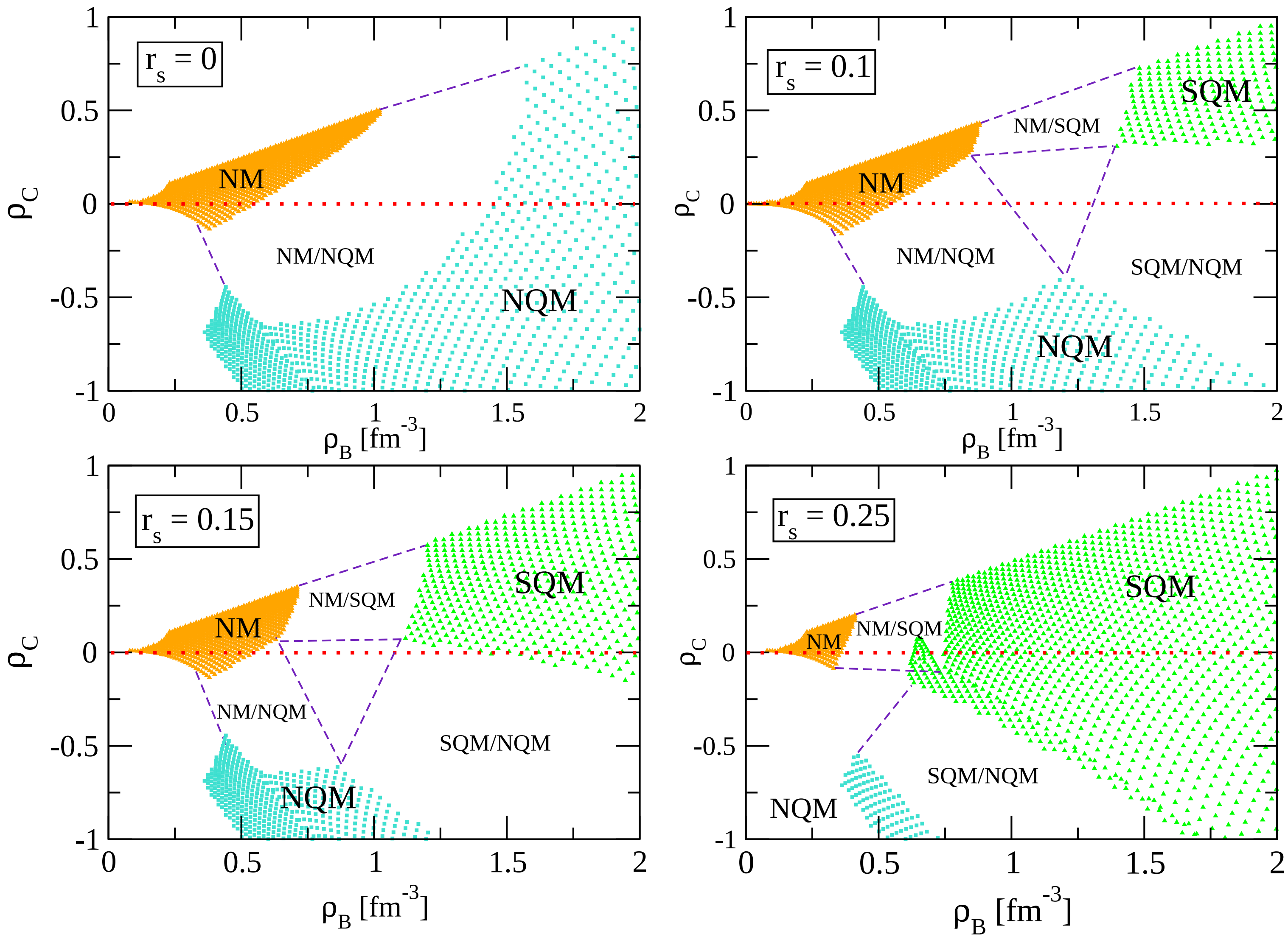
- For applications to compact stars, one must extend the description to **isospin asymmetric systems** (different numbers of protons/neutrons, or up/down quarks): Introduce a second chemical potential for isospin, and electrons in β-equilibrium.
- Look for **Phase transition NM $\rightarrow$ QM** by using Gibbs criteria:
- Draw a **phase diagram** in the plane of the 2 chemical potentials. Identify regions where **nuclear matter** (NM), **normal quark matter** (NQM), or **superconducting quark matter** (SQM, or "2SC-phase") have the largest pressure.
- Along the **phase boundaries**: Determine the volume fractions of the two phases so as to get a charge neutral mixed phase.

  (See: N. Glendenning, Phys, Rev. **D 46** (1992) 1274.)
- Use the resulting charge neutral equation of state as input in the **Tolman-Oppenheimer-Volkoff (TOV) equation** to calculate compact stars.

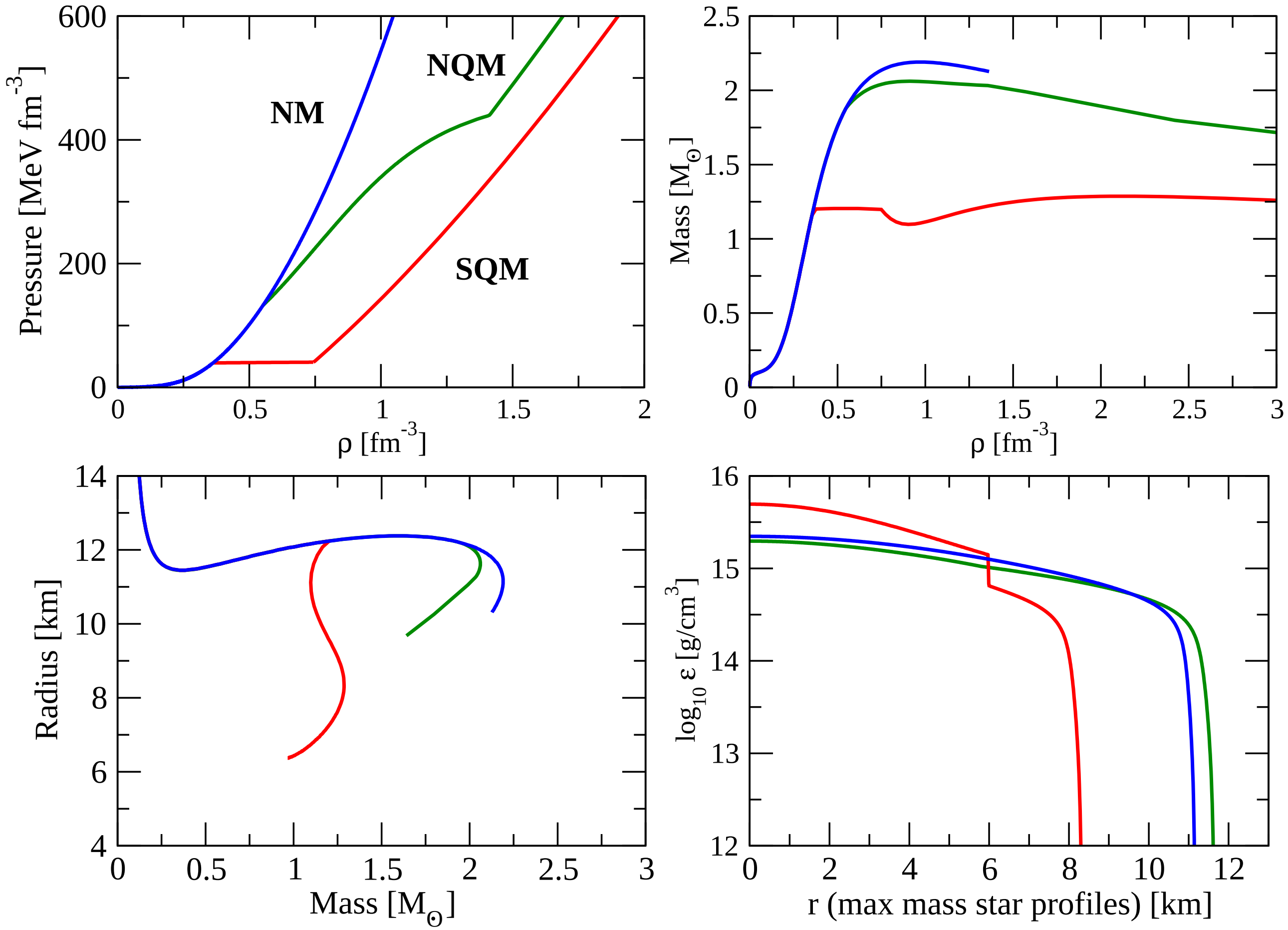
Phase diagrams (1): Fig. 23



Phase diagrams (2): Fig. 24



Compact stars: Case $r_s = 0.25$ *(Fig.25)*



Compact stars

- The pressure is almost constant in the mixed phase (similar to naive Maxwell construction).
- There are stable (almost-)neutron stars with $M < 1.25\,M_\odot$ and $\rho_c < 0.7\,\mathrm{fm}^{-3}$, which may have a small mixed phase core.
- There are stable stars made almost of SQM with $1.1\,M_\odot < M < 1.3\,M_\odot$ and $\rho_c \simeq 1 - 2\,\mathrm{fm}^{-3}$.

  For example, the maximum mass star has a radius of 8.2 km, and the SQM phase is realized within 6.0 km.

End of this lecture

The good things about the NJl model are:

- It is simple.
- We can describe non-perturbative effects (like bound states) in terms of quarks.
- We can extend it to finite density, temperature, and finite nuclei.
- We can use this model to make predictions.

Comments on figures

- Figs.11, 12, 13: See *W. Bentz et al, Nucl. Phys.* **A 720** *(2003), p. 95; Figs. 1, 2, 3.* The proper time cut-off is used here ($\Lambda_{\mathrm{UV}} = 0.64$ GeV, $\Lambda_{\mathrm{IR}} = 0.2$ GeV). The 4-Fermi coupling constants are $G_\pi = 19.6$ GeV, $r_s \equiv G_s/G_\pi = 0.51$, $r_\omega \equiv G_\omega/G_\pi = 0.37$.

- Fig.14: Here $\epsilon_F = 0.914$ GeV, $p_F = 0.26$ GeV is used in the expression above the figure. For the derivation of the formula and discussions, see: *H. Mineo et al, Nucl. Phys.* **A 735** *(2004), p. 482; sect.2.2*.

- Figs. 15, 16: See *H. Mineo et al, Nucl. Phys.* **A 735** *(2004), p. 482; Figs. 9, 11*. The proper time cut-off is used here ($\Lambda_{\mathrm{UV}} = 0.64$ GeV, $\Lambda_{\mathrm{IR}} = 0.2$ GeV). The effective masses of the quark, diquark and nucleon at zero density are 0.4 GeV, 0.576 GeV, 0.94 GeV, and at density $\rho = 0.16$ fm^{-3} they are 0.308 GeV, 0.413 GeV, 0.707 GeV. The Fermi energy of the nucleon is $\epsilon_F = 0.914$ GeV.

- Figs. 17, 18: See *I.C. Cloët et al, Phys. Lett.* **B 642** *(2006), p. 210; Figs. 6, 7, 9.* The calculation of the quark momentum distributions in the free nucleon includes both the scalar and axial vector diquark channels, see *I.C. Cloët et al, Phys. Lett.* **B 621** *(2005), p. 246* for details. The nucleon momentum distributions are calculated for finite nuclei in the mean field approximation. The proper time cut-off is used in all calculations ($\Lambda_{\mathrm{UV}} = 0.64$ GeV, $\Lambda_{\mathrm{IR}} = 0.2$ GeV).

- Figs. 19 - 22: See *W. Bentz et al, Nucl. Phys.* **A 720** *(2003), p. 95; Figs. 8, 13, 14.* The proper-time regularization is used in all calculations. For the parameters, see Table 1 of the paper.

- Figs. 23 - 25: See *S. Lawley et al, Phys. Lett.* **B 632** *(2006), p. 495; Figs. 1 - 3.* The proper-time regularization is used in all calculations. For the parameters, see the paper.